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SOLUTION OF THE IRREDUCIBLE CASE;

Or, to express the three roots of the general and complete equation of the third degree in finite, algebraic and really performable functions of the coëfficients of the equation, when these roots are all real and at least one of them is rational.*†

By Guido Weichold, *of Zittau, Saxony.*

Let a, b, c be the roots of the general and complete equation of the third degree:

$$x^3 + Ax^2 + Bx + C = 0; \text{ and}$$

$$\theta = \frac{-1+\sqrt{-3}}{2}, \quad \theta' = \frac{-1-\sqrt{-3}}{2}. \quad \text{Whence,}$$

$$a + \theta b + \theta' c = \frac{(2a-b-c) + (b-c)\sqrt{-3}}{2} = \rho;$$

$$bc + \theta ac + \theta' ab = \frac{(2bc-ac-ab) - a(b-c)\sqrt{-3}}{2} = \rho_1;$$

$$a + \theta' b + \theta c = \frac{(2a-b-c) - (b-c)\sqrt{-3}}{2} = \rho';$$

$$bc + \theta' ac + \theta ab = \frac{(2bc-ac-ab) + a(b-c)\sqrt{-3}}{2} = \rho'_1; \ddagger$$

then from the known relations:

$$a + b + c = -A; \quad ab + ac + bc = B; \quad abc = -C:$$

$$a = \frac{-A+\rho+\rho'}{3} = -\frac{\rho_1-\rho'_1}{\rho-\rho'} = -\frac{2B-(\rho_1+\rho'_1)}{2A+(\rho+\rho')} = -\frac{3C}{B+(\rho_1+\rho'_1)}$$

$$b = \frac{-A+\theta'\rho+\theta\rho'}{3} = -\frac{\theta'\rho_1-\theta\rho'_1}{\theta'\rho-\theta\rho'} = -\frac{2B-(\theta'\rho_1+\theta\rho'_1)}{2A+(\theta'\rho+\theta\rho')} = -\frac{3C}{B+(\theta'\rho_1+\theta\rho'_1)};$$

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* From the above enunciation it is obvious, that the so-called trigonometrical solution of a cubic equation is no more a solution of the irreducible case than the division of an angle into three equal parts by means of a protractor, would be a solution of the problem of the trisection of an angle, as it in reality amounts to no more than the copying of the roots from tables, where their *approximate* values are indirectly given.

† In the case where the three roots are all irrational, it would be as absurd to require these roots to be determined with numerical exactness, as it would be to require a numerically exact root of a number which is an imperfect power of the degree of the root to be extracted. Moreover, the determination of the approximate values of such roots has nothing to do with the problem of the irreducible case, as the operations indicated in Cardan's formula become performable when treated by approximation (expansion into series).

‡ Consequently ρ and ρ', as well as ρ_1 and ρ'_1, are *conjugate* complex functions of a, b, c, *i. e.*, quantities consisting of two parts, one of which is real and the other imaginary.

$$c = \frac{-A+\theta\rho+\theta'\rho'}{3} = -\frac{\theta\rho_1-\theta'\rho'_1}{\theta\rho-\theta'\rho'} = -\frac{2B-(\theta\rho_1+\theta'\rho'_1)}{2A+(\theta\rho+\theta'\rho')}, = -\frac{3C}{B+(\theta\rho_1+\theta'\rho'_1)};$$

or written collectively:

$$\left.\begin{matrix}a\\b\\c\end{matrix}\right\} = \frac{-A+\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho+\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'}{3} = -\frac{\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho_1-\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'_1}{\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho-\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'} = -\frac{2B-\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho_1-\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'_1}{2A+\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho+\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'}$$

$$= -\frac{3C}{B+\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho'+\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'_1}.$$

Now the four auxiliary quantities $\rho, \rho', \rho_1, \rho'_1$ can be expressed in terms of the coëfficients A, B, C of the proposed equation by means of the following seven relations:

1°. $\rho\rho' = \theta\rho.\theta'\rho' = \theta'\rho.\theta\rho' = (a+\theta b+\theta' c)(a+\theta' b+\theta c) = (b+\theta a+\theta' c)(b+\theta' a+\theta c) = (c+\theta a+\theta' b)(c+\theta' a+\theta b) = \frac{1}{4}\{[(a-b)+(a-c)]^2+3(b-c)^2\} = \frac{1}{4}\{[(b-c)-(a-b)]^2+3(a-c)^2\} = \frac{1}{4}\{[(a-c)+(b-c)]^2+3(a-b)^2\}^* = (a-b)^2-(a-b)(a-c)+(a-c)^2 = (b-c)^2+(b-c)(a-b)+(a-b)^2 = (a-c)^2-(a-c)(b-c)+(b-c)^2 = (a^2+b^2+c^2)-(ab+ac+bc) = A^2-3B = N$, for brevity;

2°. $\rho_1\rho'_1 = \theta\rho_1.\theta'\rho'_1 = \theta'\rho_1.\theta\rho'_1 = (bc+\theta ab+\theta' ac)(bc+\theta' ab+\theta ac) = (ac+\theta ab+\theta' bc)(ac+\theta' ab+\theta bc) = (ab+\theta ac+\theta' bc)(ab+\theta' ac+\theta bc) = \frac{1}{4}\{[b(a-c)+c(a-b)]^2+3a^2(b-c)^2\} = \frac{1}{4}\{[c(a-b)-a(b-c)]^2+3b^2(a-c)^2\} = \frac{1}{4}\{[a(b-c)+b(a-c)]^2+3c^2(a-b)^2\}^* = b^2(a-c)^2-bc(a-b)(a-c)+c^2(a-b)^2 = c^2(a-b)^2+ac(a-b)(b-c)+a^2(b-c)^2 = a^2(b-c)^2-ab(b-c)(a-c)+b^2(a-c)^2 = a^2b^2+a^2c^2+b^2c^2-abc(a+b+c) = B^2-3AC = N'$;

3°. $\rho\rho'_1+\rho'\rho_1 = \theta\rho.\theta'\rho'_1+\theta'\rho'.\theta\rho_1 = \theta'\rho.\theta\rho'_1+\theta\rho'.\theta'\rho_1 = (a+\theta b+\theta' c)(bc+\theta' ac+\theta ab)+(a+\theta' b+\theta c)(bc+\theta ac+\theta' ab) = (c+\theta a+\theta' b)(ab+\theta' bc+\theta ac)+(c+\theta' a+\theta b)(ab+\theta bc+\theta' ac) = (b+\theta c+\theta' a)(ac+\theta' ab+\theta bc)+(b+\theta' c+\theta a)(ac+\theta ab+\theta' bc) = 6abc+(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2)(\theta+\theta') = 9abc-(ab+ac+bc)(a+b+c) = AB-9C = P$;

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* These expressions show that N and N' can be reduced to the form: $\alpha^2+3\beta^2 = (\alpha+\beta\sqrt{-3})(\alpha-\beta\sqrt{-3})$, when a, b, c are real and integral, and to the form: $\frac{\alpha^2+3\beta^2}{\gamma^2} = \frac{(\alpha+\beta\sqrt{-3})}{\gamma}\frac{(\alpha-\beta\sqrt{-3})}{\gamma}$, when a, b, c are real and fractional, α, β, γ denoting integers.

4°. $\rho^3 + \rho'^3 = (\rho + \rho')(\theta\rho + \theta'\rho')(\theta'\rho + \theta\rho') = (2a - b - c)(2b - a - c)(2c - a - b) = \{(a-b) + (a-c)\}\{(a-c) + (b-c)\}\{(a-b) - (b-c)\} = 2(a^3 + b^3 + c^2) - 3(ab + ac + bc)(a + b + c) + 27\,abc = -\,2A^3 + 9\,AB - 27\,C = 3P - 2AN = M;$

5°. $\rho_1^3 + \rho'^3_1 = (\rho_1 + \rho'_1)(\theta\rho_1 + \theta'\rho'_1)(\theta'\rho_1 + \theta\rho'_1) = (2bc - ab - ac)(2ac - ab - bc)(2ab - ac - bc) = \{b(a - c) + c(a - b)\}\{a(b - c) - c(a - b)\}\{a(b - c) + b(a - c)\} = 2(a^3b^3 + a^3c^3 + b^3c^3) - 3abc(ab + ac + bc)(a + b + c) + 27a^2b^2c^2 = 2B^3 - 9ABC + 27C^2 = -3CP + 2BN' = M';$

6°. $\rho^2\rho_1 + \rho'^2\rho'_1 = (\theta\rho)^2(\theta\rho_1) + (\theta'\rho')^2(\theta'\rho'_1) = (\theta'\rho)^2(\theta'\rho_1) + (\theta\rho')^2(\theta\rho'_1) = (a + \theta b + \theta' c)^2(bc + \theta' ab + \theta ac) + (a + \theta' b + \theta c)^2(bc + \theta ab + \theta' ac) = (c + \theta a + \theta' b)^2(ab + \theta' ac + \theta bc) + (c + \theta' a + \theta b)^2(ab + \theta ac + \theta' bc) = (b + \theta c + \theta' a)^2(ac + \theta' bc + \theta ab) + (b + \theta' c + \theta a)^2(ac + \theta bc + \theta' ab) = -\,abc(a + b + c) + 4(a^2b^2 + a^2c^2 + b^2c^2) - (a^2 + b^2 + c^2)(ab + ac + bc) = -9AC + 6B^2 - A^2B = 3N' - BN = AP - 2BN;$

7°. $\rho\rho_1^2 + \rho'\rho'^2_1 = (\theta\rho)(\theta\rho_1)^2 + (\theta'\rho')(\theta'\rho'_1)^2 = (\theta'\rho)(\theta'\rho_1)^2 + (\theta\rho')(\theta\rho'_1)^2 = (a + \theta b + \theta' c)(bc + \theta ac + \theta' ab)^2 + (a + \theta' b + \theta c)(bc + \theta' ac + \theta ab)^2 = (c + \theta a + \theta' b)(ab + \theta bc + \theta' ac)^2 + (c + \theta' a + \theta b)(ab + \theta' bc + \theta ac)^2 = (b + \theta c + \theta' a)(ac + \theta' bc + \theta ab)^2 + (b + \theta' c + \theta a)(ac + \theta bc + \theta' ab)^2 = -\,abc(ab + ac + bc) + 4abc(a^2 + b^2 + c^2) - (a + b + c)(a^2b^2 + a^2c^2 + b^2c^2) = 9BC - 6A^2C + AB^2 = AN' - 3CN = 2AN' - BP;$

From the foregoing seven relations follow:

1°. $\rho\rho_1 - \rho'\rho_1 = (a + \theta b + \theta' c)(bc + \theta' ac + \theta ab) - (a + \theta' b + \theta c)(bc + \theta ac + \theta' ab) = (a - b)(a - c)(b - c)\sqrt{-3} = \sqrt{(\rho\rho'_1 + \rho'\rho_1)^2 - 4\rho\rho'\rho_1\rho'_1} = \sqrt{P^2 - 4NN} = S\sqrt{-3}$, putting $\sqrt{\frac{4NN' - P^2}{3}} = S;$

2°. $\rho^3 - \rho'^3 = (\rho - \rho')(\theta\rho + \theta'\rho')(\theta'\rho - \theta\rho') = 3(a - b)(a - c)(b - c)\sqrt{-3} = \sqrt{(\rho^3 + \rho'^3)^2 - 4\rho^3\rho'^3} = \sqrt{M^2 - 4N^3} = \sqrt{(3P - 2AN)^2 - 4N^3} = \sqrt{9P^2 - 12APN + 4A^2N^2 - 4N^3} = \sqrt{9P^2 - 4N(N^2 - A^2N + 3AP)} = \sqrt{9P^2 - 4N.9N'} = 3\sqrt{P^2 - 4NN'} = 3S\sqrt{-3};$

3°. $\rho_1^3 - \rho'^3_1 = (\rho_1 - \rho'_1)(\theta\rho_1 - \theta'\rho'_1)(\theta'\rho_1 - \theta\rho'_1) = -3abc(a - b)(a - c)(b - c)\sqrt{-3} = \sqrt{(\rho_1^3 + \rho'^3_1)^2 - 4\rho_1^3\rho'^3_1} = \sqrt{M'^2 - 4N'^3} = \sqrt{(-3CP + 2BN')^2 - 4N'^3} = \sqrt{9C^2P^2 - 12BCN'P + 4B^2N'^2 - 4N'^3} = \sqrt{9C^2P^2 - 4N'(N'^2 - B^2N' + 3BCP)} = \sqrt{9C^2P^2 - 4N.9C^2N} = 3C\sqrt{P^2 - 4NN'} = 3CS\sqrt{-3};$

4°. $\rho^2\rho_1 - \rho'^2\rho'_1 = (a + \theta b + \theta' c)^2(bc + \theta' ab + \theta ac) - (a + \theta' b + \theta c)^2(bc + \theta ab +$
$\theta' ac) = (a + b + c)(a - b)(a - c)(b - c)\sqrt{-3} = \sqrt{(\rho^2\rho_1 + \rho'^2\rho'_1)^2 - 4\rho^2\rho'^2\rho_1\rho'_1}$
$= \sqrt{(AP - 2BN)^2 - 4N^2N'} = \sqrt{A^2P^2 - 4ABPN + 4B^2N^2 - 4N^2N'}$
$= \sqrt{A^2P^2 - 4N(NN' - B^2N + ABP)} = \sqrt{A^2P^2 - 4N.A^2N'}$
$= A\sqrt{P^2 - 4NN'} = AS\sqrt{-3}$;

5°. $\rho\rho_1^2 - \rho'\rho_1'^2 = (a + \theta b + \theta' c)(bc + \theta' ab + \theta ac)^2 - (a + \theta' b + \theta c)(bc + \theta ab + \theta' ac)^2$
$= (ab + ac + bc)(a - b)(a - c)(b - c)\sqrt{-3} = \sqrt{(\rho\rho_1^2 + \rho'\rho_1'^2)^2 - 4\rho\rho'\rho_1^2\rho_1'^2}$
$= \sqrt{(2AN' - BP)^2 - 4NN'^2} = \sqrt{4A^2N'^2 - 4ABPN' + B^2P^2 - 4NN'^2}$
$= \sqrt{B^2P^2 - 4N'(NN' - A^2N' + ABP)} = \sqrt{B^2P^2 - 4N'.B^2N}$
$= B\sqrt{P^2 - 4NN'} = BS\sqrt{-3}$;

and, moreover:

$\rho\rho'_1 = \frac{1}{2}\{(\rho\rho'_1 + \rho'\rho_1) + (\rho\rho'_1 - \rho'\rho_1)\} = \frac{1}{2}\{P + \sqrt{P^2 - 4NN'}\} = \frac{1}{2}\{P + S\sqrt{-3}\}$;

$\rho'\rho_1 = \frac{1}{2}\{(\rho\rho'_1 + \rho'\rho_1) - (\rho\rho'_1 - \rho'\rho_1)\} = \frac{1}{2}\{P - \sqrt{P^2 - 4NN'}\} = \frac{1}{2}\{P - S\sqrt{-3}\}$;

$\rho^3 = \frac{1}{2}\{(\rho^3 + \rho'^3) + (\rho^3 - \rho'^3)\} = \frac{1}{2}\{M + \sqrt{M^2 - 4N^3}\} = \frac{1}{2}\{3P - 2AN +$
$3S\sqrt{-3}\}$;

$\rho'^3 = \frac{1}{2}\{(\rho^3 + \rho'^3) - (\rho^3 - \rho'^3)\} = \frac{1}{2}\{M - \sqrt{M^2 - 4N^3}\} = \frac{1}{2}\{3P - 2AN -$
$3S\sqrt{-3}\}$;

$\rho_1^3 = \frac{1}{2}\{(\rho_1^3 + \rho_1'^3) + (\rho_1^3 - \rho_1'^3)\} = \frac{1}{2}\{M' + \sqrt{M'^2 - 4N'^3}\} = \frac{1}{2}\{-3CP +$
$2BN' + 3CS\sqrt{-3}\}$;

$\rho_1'^3 = \frac{1}{2}\{(\rho_1^3 + \rho_1'^3) - (\rho_1^3 - \rho_1'^3)\} = \frac{1}{2}\{M' - \sqrt{M'^2 - 4N'^3}\} = \frac{1}{2}\{-3CP +$
$2BN' - 3CS\sqrt{-3}\}$;

$\rho^2\rho_1 = \frac{1}{2}\{(\rho^2\rho_1 + \rho'^2\rho'_1) + (\rho^2\rho_1 - \rho'^2\rho'_1)\} = \frac{1}{2}\{3N' - BN + A\sqrt{P^2 - 4NN'}\} =$
$\frac{1}{2}\{AP - 2BN + AS\sqrt{-3}\}$;

$\rho'^2\rho'_1 = \frac{1}{2}\{(\rho^2\rho_1 + \rho'^2\rho'_1) - (\rho^2\rho_1 - \rho'^2\rho'_1)\} = \frac{1}{2}\{3N' - BN - A\sqrt{P^2 - 4NN'}\}$
$= \frac{1}{2}\{AP - 2BN - AS\sqrt{-3}\}$;

$\rho\rho_1^2 = \frac{1}{2}\{(\rho\rho_1^2 + \rho'\rho_1'^2) + (\rho\rho_1^2 - \rho'\rho_1'^2)\} = \frac{1}{2}\{AN' - 3CN + B\sqrt{P^2 - 4NN'}\} =$
$\frac{1}{2}\{2AN' - BP + BS\sqrt{-3}\}$;

$\rho'\rho_1'^2 = \frac{1}{2}\{(\rho\rho_1^2 + \rho'\rho_1'^2) - (\rho\rho_1^2 - \rho'\rho_1'^2)\} = \frac{1}{2}\{AN' - 3CN - B\sqrt{P^2 - 4NN'}\} =$
$\frac{1}{2}\{2AN' - BP - BS\sqrt{-3}\}$.

Now, as these values of $\rho\rho'_1$, $\rho'\rho_1$, ρ^3, ρ'^3, ρ_1^3, $\rho_1'^3$, $\rho^2\rho_1$, $\rho'^2\rho'_1$, $\rho\rho_1^2$, $\rho'\rho_1'^2$, are complex (see note, page 32) when a, b, c are simultaneously real, as appears from the inspection of $S = (a - b)(a - c)(b - c) = \sqrt{\frac{4NN' - P^2}{3}}$, showing that S

is real and rational when a, b, c are real and rational, the determination of ρ, ρ', ρ_1, ρ'_1, by the extraction of the cube root of $\rho^3 = \frac{1}{2}\{M + 3S\sqrt{-3}\}$; $\rho'^3 = \frac{1}{2}\{M - 3S\sqrt{-3}\}$; $\rho_1^3 = \frac{1}{2}\{M' + 3CS\sqrt{-3}\}$; $\rho_1'^3 = \frac{1}{2}\{M' - 3CS\sqrt{-3}\}$, would involve the irreducible case. But the values of the four auxiliary quantities ρ, ρ', ρ_1, ρ'_1, can be found without the operation of extracting the cube root of ρ^3, ρ'^3, ρ_1^3, $\rho_1'^3$, by the determination, in various ways, of the factors respectively common. Thus:

I. to $\rho\rho'$ and $\rho\rho'_1$, viz. ρ; to $\rho_1\rho'_1$ and $\rho'\rho_1$, viz. ρ_1;
“ $\rho\rho'$ “ $\rho'\rho_1$, “ ρ'; “ $\rho_1\rho'_1$ “ $\rho\rho'_1$, “ ρ'_1;
II. to $\rho\rho'$ and ρ^3, viz. ρ; to $\rho_1\rho'_1$ and ρ_1^3, viz. ρ_1;
“ $\rho\rho'$ “ ρ'^3, “ ρ'; “ $\rho_1\rho'_1$ “ $\rho_1'^3$, “ ρ'_1;
III. to $\rho\rho'$ and $\rho^2\rho_1$, viz. ρ; to $\rho_1\rho'_1$ and $\rho^2\rho_1$, viz. ρ_1;
“ $\rho\rho'$ “ $\rho'^2\rho'_1$, “ ρ'; “ $\rho_1\rho'_1$ “ $\rho'^2\rho'_1$, “ ρ'_1;
IV. to $\rho\rho'$ and $\rho\rho_1^2$, viz. ρ; to $\rho_1\rho'_1$ and $\rho\rho_1^2$, viz. ρ_1;
“ $\rho\rho'$ “ $\rho'\rho_1'^2$, “ ρ'; “ $\rho_1\rho'_1$ “ $\rho'\rho_1'^2$, “ ρ'_1;
V. to $\rho\rho'_1$ and $\rho^2\rho_1$, viz. ρ; to $\rho'\rho_1$ and $\rho^2\rho_1$, viz. ρ_1;
“ $\rho'\rho_1$ “ $\rho'^2\rho'_1$, “ ρ'; “ $\rho\rho'_1$ “ $\rho'^2\rho'_1$, “ ρ'_1;
VI. to $\rho\rho'_1$ and $\rho\rho_1^2$, viz. ρ; to $\rho'\rho_1$ and $\rho\rho_1^2$, viz. ρ_1;
“ $\rho'\rho_1$ “ $\rho'\rho_1'^2$, “ ρ'; “ $\rho\rho'_1$ “ $\rho'\rho_1'^2$, “ ρ'_1;
VII. to $\rho\rho'_1$ and ρ^3, viz. ρ; to $\rho'\rho_1$ and ρ_1^3, viz. ρ_1;
“ $\rho'\rho_1$ “ ρ'^3, “ ρ'; “ $\rho\rho'_1$ “ $\rho_1'^3$, “ ρ'_1;
VIII. to $\rho\rho_1^2$ and ρ^3, viz. ρ; to $\rho\rho_1^2$ and ρ_1^3, viz. ρ_1^2, whence $\rho_1 = \sqrt{\overline{\rho\rho_1^2 \mid \rho_1^3}}$; *
“ $\rho'_1\rho_1'^2$ “ ρ'^3, “ ρ'; “ $\rho'\rho_1'^2$ “ $\rho_1'^3$, “ $\rho_1'^2$, “ $\rho'_1 = \sqrt{\overline{\rho'\rho_1'^2 \mid \rho_1'^3}}$;
IX. to $\rho^2\rho_1$ and ρ^3, viz. ρ^2, whence $\rho = \sqrt{\overline{\rho^2\rho_1 \mid \rho^3}}$; to $\rho^2\rho_1$ and ρ_1^3, viz. ρ_1;
“ $\rho'^2\rho'_1$ “ ρ'^3, “ ρ'^2, “ $\rho' = \sqrt{\overline{\rho'^2\rho'_1 \mid \rho'^3}}$; to $\rho'^2\rho'_1$ and $\rho_1'^3$, viz. ρ'_1;
X. to $\rho^2\rho_1$ and $\rho\rho_1^2$, viz. $\rho\rho_1$, whence $\rho = \dfrac{\rho^2\rho_1}{\overline{\rho^2\rho_1 \mid \rho\rho_1^2}}$ and $\rho_1 = \dfrac{\rho\rho_1^2}{\overline{\rho^2\rho_1 \mid \rho\rho_1^2}}$;
“ $\rho'^2\rho'_1$ “ $\rho'\rho_1'^2$ “ $\rho'\rho'_1$, “ $\rho' = \dfrac{\rho'^2\rho'_1}{\overline{\rho'^2\rho' \mid \rho'\rho_1'^2}}$ “ $\rho'_1 = \dfrac{\rho'\rho_1'^2}{\overline{\rho'^2\rho'_1 \mid \rho'\rho_1'^2}}$;

and satisfying at the same time the above seven relations.

Therefore, by means of the sign $\overline{\mid}$, the roots a, b, c can be symbolically represented as follows:

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* The sign $\overline{\mid}$ denotes the factor in question common to the quantities on either side of it.

$$\left.\begin{matrix}a\\b\\c\end{matrix}\right\} = \frac{-A + \left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho + \left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'}{3} = \frac{-A + \left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\overline{\rho\rho'\,|\,\rho^{3}} + \left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\overline{\rho\rho'\,|\,\rho'^{3}}}{3}$$

$$= -\frac{\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho_1 - \left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'_1}{\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho - \left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'} = -\frac{\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\overline{\rho_1\rho'_1\,|\,\rho_1^{3}} - \left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\overline{\rho_1\rho'_1\,|\,\rho_1'^{3}}}{\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\overline{\rho\rho'\,|\,\rho^{3}} - \left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\overline{\rho\rho'\,|\,\rho_1^{3}}}$$

$$= -\frac{2B - \left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho_1 - \left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'_1}{2A + \left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho + \left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'} = -\frac{2B - \left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\overline{\rho_1\rho'_1\,|\,\rho_1^{3}} - \left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\overline{\rho_1\rho'_1\,|\,\rho_1'^{3}}}{2A + \left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\overline{\rho\rho'\,|\,\rho^{3}} + \left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\overline{\rho\rho'\,|\,\rho'^{3}}}$$

$$= -\frac{3C}{B + \left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho_1 + \left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'_1} = -\frac{3C}{B + \left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\overline{\rho_1\rho'_1\,|\,\rho_1^{3}} + \left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\overline{\rho_1\rho'_1\,|\,\rho_1'^{3}}}$$

and so on, by using for the values of ρ, ρ', ρ_1, ρ'_1 the other combinations specified above for the determination of the common factor in question.

Or, writing the roots in terms of the coëfficients A, B, C:

$$\left.\begin{matrix}a\\b\\c\end{matrix}\right\} = \tfrac{1}{3}\left\{-A + \left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\overline{A^2 - 3B\left|\frac{AB - 9C + \sqrt{(AB-9C)^2 - 4(A^2-3B)(B^2-3AC)}}{2}\right.}\right.$$

$$\left. + \left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\overline{A^2 - 3B\left|\frac{AB - 9C - \sqrt{(AB-9C)^2 - 4(A^2-3B)(B^2-3AC)}}{2}\right.}\right\}$$

etc.

The determination of the above common factor can be effected, as may be seen from the annexed numerical examples, by means of *a limited number of essentially algebraic operations;* that is to say, ρ, ρ', ρ_1, ρ'_1 are finite, algebraic and really performable functions of the coëfficients A, B, C of the proposed equation, *q. e. f.*

COROLLARY.

The expressions: $a = -\dfrac{\rho_1 - \rho'_1}{\rho - \rho'}$; $b = -\dfrac{\theta'\rho_1 - \theta\rho'_1}{\theta'\rho - \theta\rho'} = -\dfrac{(\rho_1 - \rho'_1) + (\rho_1 + \rho'_1)\sqrt{-3}}{(\rho - \rho') + (\rho + \rho')\sqrt{-3}}$,

$$c = -\frac{\theta\rho_1 - \theta'\rho'_1}{\theta\rho - \theta'\rho'} = -\frac{(\rho_1 - \rho'_1) - (\rho_1 + \rho'_1)\sqrt{-3}}{(\rho - \rho') - (\rho + \rho')\sqrt{-3}};$$

$\rho_1 + \rho'_1 = \dfrac{P(\rho + \rho') - (\rho - \rho') S\sqrt{-3}}{2N}$; $\rho_1 - \rho'_1 = \dfrac{P(\rho - \rho') - (\rho + \rho') S\sqrt{-3}}{2N'}$; (the last two resulting from the combination of: $\rho\rho' = N$; $\rho_1\rho'_1 = N'$; $\rho\rho'_1 = \dfrac{P + S\sqrt{-3}}{2}$; $\rho'\rho_1 = \dfrac{P - S\sqrt{-3}}{2}$) furnish, after the elimination of a, $\rho + \rho'$, $\rho - \rho'$, $\rho_1 + \rho'_1$, $\rho_1 - \rho'_1$ on the one hand, and of b, $\rho + \rho'$, $\rho - \rho'$, $\rho_1 + \rho'_1$, $\rho_1 - \rho'_1$ on the other, the two equations:

$$b = -\frac{2N' + (P + S)\,c}{2Nc + (P - S)}; \quad a = -\frac{2N' + (P - S)\,c}{2Nc + (P + S)};$$

or, written collectively, the following formula:

$$\left.\begin{matrix} a \\ b \end{matrix}\right\} = -\frac{2N' + (P \pm S)\,c}{2Nc + (P \mp S)};$$

by means of which, when one root of the proposed equation is known, the remaining two may be found in terms of the known one and the quantities N, N', P and S, which have already been calculated for the determination of the first root.

SCHOLIUM.

The following considerations will show that algebra itself points to the foregoing solution of the irreducible case:

I. The elimination of three of the four quantities ρ, ρ', ρ_1, ρ'_1, of ρ', ρ_1, ρ'_1 for instance, from the four simultaneous equations above obtained:

$$\frac{-A + \rho + \rho'}{3} = -\frac{2B - (\rho_1 + \rho'_1)}{2A + \rho + \rho'}; \quad \rho\rho' = N; \quad \rho_1\rho'_1 = N'; \quad \rho\rho'_1 + \rho'\rho_1 = P;$$

leads to the first equation in ρ:

$$2N\rho^4 - [3P - 2AN - 3S\sqrt{-3}]\,\rho^3 - N[3P - 2AN + 3S\sqrt{-3}]\rho + 2N^3 = 0;$$

which becomes decomposable into a product of two factors by putting in the last term for $N^3 = \rho^3\rho'^3$ its equivalent:

$$\rho^3\rho'^3 = \frac{3P - 2AN + 3S\sqrt{-3}}{2} \cdot \frac{3P - 2AN - 3S\sqrt{-3}}{2}, \text{ viz:}$$

$$\left\{ 2\rho^3 - (3P - 2AN + 3S\sqrt{-3}) \right\} \left\{ N\rho - \frac{3P - 2AN - 3S\sqrt{-3}}{2} \right\} = 0;$$

which can be satisfied by putting:

either $2\rho^3 - (3P - 2AN + 3S\sqrt{-3}) = 0$; or $N\rho - \frac{3P - 2AN - 3S\sqrt{-3}}{2} = 0$;

whence $\rho^3 = \frac{3P - 2AN + 3S\sqrt{-3}}{2}$; and $\rho = \frac{3P - 2AN - 3S\sqrt{-3}}{2N} = \frac{\rho'^3}{\rho\rho'}$.

The value of ρ^3 furnished by the first of these equations coincides with that previously found; the value of ρ furnished by the second, does not satisfy the proposed equation, if the quotient of $\frac{3P - 2AN - 3S\sqrt{-3}}{2} = \rho'^3$ divided by $N = \rho\rho'$ is taken in the ordinary sense, but if taken in the more general sense of *the factor common to the dividend and divisor*, it furnishes the conjugate value to ρ. This interpretation is vindicated by the fact that, as there does not yet exist any sign to denote a factor common to two quantities, that operation cannot manifest itself as the result of other operations, as is the case with addition, subtraction, etc. It would, therefore, be of advantage to add to the notation of the elementary operations, a sign to denote such a common factor, by the adoption of which other important results might likely be obtained.

II. The fact that, though it is impossible to extract the cube root of the values of ρ^3, ρ'^3, ρ_1^3, ρ'^3_1 algebraically under finite form, yet the extraction of the cube root of the products $\rho^3\rho'^3$, $\rho_1^3\rho'^3_1$, $\rho^3\rho'^3_1$, $\rho'^3\rho_1^3$, $\rho^6\rho_1^3$, $\rho'^6\rho'^3_1$, $\rho^3\rho_1^6$, $\rho'^3\rho'^6_1$ can be effected in the most general manner, as the values of these cube roots are respectively:

$$\rho\rho' = N;\ \rho_1\rho'_1 = N';\ \rho\rho'_1 = \frac{P + S\sqrt{-3}}{2};\ \rho'\rho_1 = \frac{P - S\sqrt{-3}}{2};$$

$$\rho^2\rho_1 = \frac{3N' - BN + AS\sqrt{-3}}{2};\ \rho'^2\rho'_1 = \frac{3N' - BN - AS\sqrt{-3}}{2};$$

$$\rho\rho_1^2 = \frac{AN' - 3CN + BS\sqrt{-3}}{2};\ \rho'\rho'^2_1 = \frac{AN' - 3CN - BS\sqrt{-3}}{2},$$

is another striking intimation that the determination of ρ, ρ', ρ_1, ρ'_1 is not to be effected in all cases by the extraction of the cube root of the values of ρ^3, ρ'^3, ρ_1^3, ρ'^3_1, but in certain cases by the decomposition of $\rho\rho'$, $\rho_1\rho'_1$, $\rho\rho'_1$, $\rho'\rho_1$, $\rho^2\rho_1$, $\rho'^2\rho'_1$, $\rho\rho_1^2$, $\rho'\rho'^2_1$ into their factors, *i. e.*, by the foregoing determination of common factors.

NUMERICAL EXAMPLES

Illustrating the Solution of the Irreducible Case.

I.

$$x^3 - 16x^2 + 73x - 90 = 0.$$

$$A = -16; \quad B = 73; \quad C = -90.$$

$\rho\rho' = A^2 - 3B = N = 37$; $\rho_1\rho'_1 = N' = 1009$; $\rho\rho'_1 + \rho'\rho_1 = AB - 9C = P = -358$; $\dfrac{\rho\rho'_1 - \rho'\rho_1}{\sqrt{-3}} = (a-b)(a-c)(b-c) = \sqrt{\dfrac{4NN' - P^2}{3}} = S = 84$;

$\rho^3 + \rho'^3 = 3P - 2AN = M = 110$; $\rho_1^3 + \rho'^3_1 = 2BN' - 3CP = M' = 50654$;

$\rho^2\rho_1 + \rho'^2\rho'_1 = 3N' - BN = 326$; $\rho\rho_1^2 + \rho'\rho'^2_1 = AN' - 3CN = -6154$;

$\rho\rho'_1 = \dfrac{P + S\sqrt{-3}}{2} = -179 + 42\sqrt{-3}$; $\rho'\rho_1 = \dfrac{P - S\sqrt{-3}}{2} = -179 - 42\sqrt{-3}$;

$\rho^3 = \dfrac{M + 3S\sqrt{-3}}{2} = 55 + 126\sqrt{-3}$; $\rho'^3 = \dfrac{M - 3S\sqrt{-3}}{2} = 55 - 126\sqrt{-3}$;

$\rho_1^3 = \dfrac{M' + 3CS\sqrt{-3}}{2} = 25327 - 11340\sqrt{-3}$; $\rho'^3_1 = \dfrac{M' - 3CS\sqrt{-3}}{2} = 25327 + 11340\sqrt{-3}$; $\rho^2\rho_1 = \dfrac{3N' - BN + AS\sqrt{-3}}{2} = 163 - 672\sqrt{-3}$;

$\rho'^2\rho'_1 = \dfrac{3N' - BN - AS\sqrt{-3}}{2} = 163 + 672\sqrt{-3}$; $\rho\rho_1^2 = -3077 + 3066\sqrt{-3}$;

$\rho'\rho'^2_1 = -3077 - 3066\sqrt{-3}$.

S being real and rational shows that a, b, c are real and rational, and that the irreducible case occurs here. Therefore ρ, ρ', ρ_1, ρ'_1 cannot be determined otherwise than by the process of "common factors." But as it would be too laborious to exhaust all the possible ways of determining these quantities by that process, the following three combinations are deemed sufficient to illustrate that method.

Determination of the Factor Common:

1. To $\rho\rho' = 37$ and $\rho\rho'_1 = -179 + 42\sqrt{-3}$, viz. ρ:

$$\text{First operation: } \overset{\text{dividend}}{-179 + 42\sqrt{-3}} : \overset{\text{divisor}}{37} = \overset{\text{quotient}}{-5 + \sqrt{-3}} + \overset{\text{remainder}}{\frac{6 + 5\sqrt{-3}}{37}}$$

$$= \overset{\text{quotient}}{-5 + \sqrt{-3}} + \sqrt{-3}\,\frac{(5 - 2\sqrt{-3})}{37}$$

second operation: $\overset{\text{former divisor}}{37} : \overset{\text{simplified former rem.}}{5 - 2\sqrt{-3}} = \overset{\text{hypothetical quotient}}{\alpha + \beta\sqrt{-3}}$;*

$\alpha = 5$; $\beta = 2$; *i. e.* $37 : 5 - 2\sqrt{-3} = 5 + 2\sqrt{-3}$.

Therefore: $\overline{-179 + 42\sqrt{-3}\,|\,37} = 5 - 2\sqrt{-3}$; hence

$\rho\rho' = (5 - 2\sqrt{-3})\,(5 + 2\sqrt{-3}) = (-5 + 2\sqrt{-3})\,(-5 - 2\sqrt{-3})$ and

$\rho\rho'_1 = (5 - 2\sqrt{-3})\,(-31 - 4\sqrt{-3}) = (-5 + 2\sqrt{-3})(31 + 4\sqrt{-3})$.

To decide whether $\rho = 5 - 2\sqrt{-3}$ or $= -5 + 2\sqrt{-3}$, it suffices to compare the cubes of $5 - 2\sqrt{-3}$ and $-5 + 2\sqrt{-3}$ with the above value of $\rho^3 = 55 + 126\sqrt{-3}$; thus:

$\rho = -5 + 2\sqrt{-3}$; $\rho' = -5 - 2\sqrt{-3}$; $\rho_1 = 31 - 4\sqrt{-3}$; $\rho'_1 = 31 + 4\sqrt{-3}$.

2. To $\rho\rho'_1 = -179 + 42\sqrt{-3}$ and $\rho_1\rho'_1 = 1009$, viz. ρ'_1:

First operation: $\overset{\text{div'd}}{1009} : \overset{\text{div'r}}{-179 + 42\sqrt{-3}} = \overset{\text{hypoth. quot.}}{\alpha + \beta\sqrt{-3}}$; $\alpha = -\dfrac{179}{37}$; $\beta = -\dfrac{42}{37}$; in integers approximately $\alpha = -5$; $\beta = -1$; remainder: $-12 + 31\sqrt{-3} = \sqrt{-3}\,(31 + 4\sqrt{-3})$;

second operation: $\overset{\text{former div'r}}{-179 + 4\sqrt{-3}} : \overset{\text{simplf. former rem.}}{31 + 4\sqrt{-3}} = \overset{\text{hypoth. quot.}}{\alpha + \beta\sqrt{-3}}$; $\alpha = -5$; $\beta = 2$; *i. e.* $-179 + 42\sqrt{-3} : 31 + 4\sqrt{-3} = -5 + 2\sqrt{-3}$. Therefore: $\overline{1009\,|\,-179 + 42\sqrt{-3}} = 31 + 4\sqrt{-3}$; hence $\rho\rho'_1 = (31 + 4\sqrt{-3})\,(-5 + 2\sqrt{-3}) = (-31 - 4\sqrt{-3})\,(5 - 2\sqrt{-3})$;

$\rho_1\rho'_1 = (31 - 4\sqrt{-3})(31 + 4\sqrt{-3}) = (-31 + 4\sqrt{-3})(-31 - 4\sqrt{-3})$.

From the comparison of the cubes of $-5 + 2\sqrt{-3}$ and $5 - 2\sqrt{-3}$ with the value of $\rho^3 = 55 + 126\sqrt{-3}$, we have, as before:

$\rho = -5 + 2\sqrt{-3}$; $\rho' = -5 - 2\sqrt{-3}$; $\rho_1 = 31 - 4\sqrt{-3}$; $\rho'_1 = 31 + 4\sqrt{-3}$.

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* $\dfrac{m + n\sqrt{-3}}{p + q\sqrt{-3}} = \dfrac{(m + n\sqrt{-3})(p - q\sqrt{-3})}{(p + q\sqrt{-3})(p - q\sqrt{-3}} = \dfrac{(mp + 3nq) + (np - mq)\sqrt{-3}}{p^2 + 3q^2} = \dfrac{mp + 3nq}{p^2 + 3q^2} + \left(\dfrac{np - mq}{p^2 + 3q^2}\right)\sqrt{-3}$

shows that the quotient of the division of one complex quantity by another is in general likewise a complex quantity. If, therefore, this quotient be denoted as above by $\alpha + \beta\sqrt{-3}$, the values of α and β are expressed by the two formulæ:

$$\alpha = \frac{mp + 3nq}{p^2 + 3q^2};\ \beta = \frac{np - mq}{p^2 + 3q^2},$$

by means of which α and β are calculated in the above as well as in the following examples.

3. To $\rho_1\rho'_1 = 1009$ and $\rho^2\rho_1 = 163 - 672\sqrt{-3}$, viz. ρ_1:

First operation: $1009 : 163 - 672\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = \frac{163}{1369}$; $\beta = \frac{672}{1369}$; or approximately in integers $\alpha = 1$; $\beta = 0$; remainder: $346 + 672\sqrt{-3} = 6\sqrt{-3}\,(112 - 47\sqrt{-3})$;

second operation: $163 - 672\sqrt{-3} : 112 - 47\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = \frac{112}{19}$; $\beta = \frac{-67}{19}$; or nearly $\alpha = 6$; $\beta = -3$; remainder: $-43 - 27\sqrt{-3}$;

third operation: $112 - 47\sqrt{-3} : -43 - 27\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = \frac{1}{4}$; $\beta = \frac{5}{4}$; or nearly: $\alpha = 0$; $\beta = 1$; remainder: $31 - 4\sqrt{-3}$;

fourth operation: $-43 - 27\sqrt{-3} : 31 - 4\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = -1$; $\beta = -1$; *i. e.* $-43 - 27\sqrt{-3} : 31 - 4\sqrt{-3} = -1 - \sqrt{-3}$.

Consequently: $\overline{1009 \mid 163 - 672\sqrt{-3}} = 31 - 4\sqrt{-3}$; whence: $\rho_1\rho'_1 = (31 - 4\sqrt{-3})(31 + 4\sqrt{-3}) = (-31 + 4\sqrt{-3})(-31 - 4\sqrt{-3})$; and $\rho^2\rho_1 = (13 - 20\sqrt{-3})(31 - 4\sqrt{-3}) = (-13 + 20\sqrt{-3})(-31 + 4\sqrt{-3})$. From the comparison of the cubes of $31 - 4\sqrt{-3}$ and $-31 + 4\sqrt{-3}$ with the above value of ρ_1^3, we have: $\rho_1 = 31 - 4\sqrt{-3}$; $\rho'_1 = 31 + 4\sqrt{-3}$; and from the division of the above value of $\rho\rho'_1$ by that of ρ'_1, $\rho = -5 + 2\sqrt{-3}$; $\rho' = -5 - 2\sqrt{-3}$.

Determination of the roots a, b, c:

$$a = \frac{-A + \rho + \rho'}{3} = \frac{16 - 10}{3} = 2 = -\frac{\rho_1 - \rho'_1}{\rho - \rho'} = -\frac{-8\sqrt{-3}}{+4\sqrt{-3}} = 2$$

$$= -\frac{2B - (\rho_1 + \rho'_1)}{2A + (\rho + \rho')} = -\frac{146 - 62}{-32 - 10} = 2 = \frac{-3C}{B + \rho_1 + \rho'_1} = -\frac{-270}{73 + 62} = 2;$$

$$b = \frac{-A + \theta'\rho + \theta\rho'}{3} = \frac{-2A - (\rho + \rho') - (\rho - \rho')\sqrt{-3}}{6} = \frac{32 + 10 - 4\sqrt{-3}\sqrt{-3}}{6} = 9$$

$$= -\frac{\theta'\rho_1 - \theta\rho'_1}{\theta'\rho - \theta\rho'} = -\frac{-(\rho_1 - \rho'_1) - (\rho_1 + \rho'_1)\sqrt{-3}}{-(\rho - \rho') - (\rho + \rho')\sqrt{-3}} = -\frac{8\sqrt{-3} - 62\sqrt{-3}}{-4\sqrt{-3} + 10\sqrt{-3}} = 9$$

$$= -\frac{2B - (\theta'\rho_1 + \theta\rho'_1)}{2A + (\theta'\rho + \theta\rho')} = -\frac{4B + (\rho_1 + \rho'_1) + (\rho_1 - \rho'_1)\sqrt{-3}}{4A - (\rho + \rho') - (\rho - \rho')\sqrt{-3}} = -\frac{292 + 62 + 24}{-64 + 10 + 12} = 9$$

$$= -\frac{3C}{B + \theta'\rho_1 + \theta\rho'_1} = -\frac{6C}{2B - (\rho_1 + \rho'_1) - (\rho_1 - \rho'_1)\sqrt{-3}} = -\frac{-540}{146 - 62 - 24} = 9;$$

$$c = \frac{-A+\theta\rho+\theta'\rho'}{3} = \frac{-2A-(\rho+\rho')+(\rho-\rho')\sqrt{-3}}{6} = \frac{32+10+4\sqrt{-3}\sqrt{-3}}{6} = 5$$

$$= -\frac{\theta\rho_1-\theta'\rho'_1}{\theta\rho-\theta'\rho'} = -\frac{-(\rho_1-\rho'_1)+(\rho_1+\rho'_1)\sqrt{-3}}{-(\rho-\rho')+(\rho+\rho')\sqrt{-3}} = -\frac{8\sqrt{-3}+62\sqrt{-3}}{-4\sqrt{-3}-10\sqrt{-3}} = 5$$

$$= -\frac{2B-(\theta\rho_1+\theta'\rho'_1)}{2A+(\theta\rho+\theta'\rho')} = -\frac{4B+(\rho_1+\rho'_1)-(\rho_1-\rho'_1)\sqrt{-3}}{4A-(\rho+\rho')+(\rho-\rho')\sqrt{-3}} = -\frac{292+62-24}{-64+10-12} = 5$$

$$= -\frac{3C}{B+\theta\rho_1+\theta'\rho'_1} = -\frac{6C}{2B-(\rho_1+\rho'_1)+(\rho_1-\rho'_1)\sqrt{-3}} = -\frac{-540}{146-62+24} = 5.$$

Moreover, any two of the roots can be found from the third by means of the formula given in the text: $\left.\begin{matrix} a \\ b \end{matrix}\right\} = -\frac{2N'+(P\mp S)\,c}{2Nc+(P\pm S)}$; thus:

1° from $c=5$; $a = -\frac{2018-442\cdot 5}{74\cdot 5-274} = \frac{192}{96} = 2$; and

$b = -\frac{2018-274\cdot 5}{74\cdot 5-442} = -\frac{648}{-72} = 9$;

2° from $b=9$; $a = -\frac{2018-274\cdot 9}{74\cdot 9-442} = -\frac{-448}{224} = 2$; and

$c = -\frac{2018-442\cdot 9}{74\cdot 9-274} = -\frac{-1960}{392} = 5$;

3° from $a=2$; $b = -\frac{2018-442\cdot 2}{74\cdot 2-274} = -\frac{1134}{-126} = 9$; and

$c = -\frac{2018-274\cdot 2}{74\cdot 2-442} = -\frac{1470}{-294} = 5.$

II.

$$x^3 + \frac{41}{140}x^2 - \frac{79}{14}x + \frac{429}{140} = 0;$$

$$A = \frac{41}{140};\ B = -\frac{79}{14};\ C = \frac{429}{140};$$

$\rho\rho' = A^2 - 3B = N = \frac{333481}{19600}$; $\rho_1\rho'_1 = B^2 - 3AC = N' = \frac{571333}{19600}$; $\rho\rho'_1 + \rho'\rho_1 = AB - 9C = P = -\frac{572930}{19600}$; $\frac{\rho\rho'_1-\rho'\rho_1}{\sqrt{-3}} = \sqrt{\frac{4NN'-P^2}{3}} = S = \frac{380292}{19600}$;

$$\rho\rho'_1 = \frac{P + S\sqrt{-3}}{2} = \frac{-286465 + 190146\sqrt{-3}}{19600}; \; \rho\rho_1 = \frac{P - S\sqrt{-3}}{2}$$

$$= \frac{-286465 - 190146\sqrt{-3}}{19600};$$

$$\rho^3 = \frac{3P - 2AN + 3S\sqrt{-3}}{2} = \frac{-13398021 + 79861320\sqrt{-3}}{2744000};$$

$$\rho'^3 = \frac{3P - 2AN - 3S\sqrt{-3}}{2} = \frac{-13398021 - 79861320\sqrt{-3}}{2744000};$$

$$\rho_1^3 = \frac{-82672615 + 244717902\sqrt{-3}}{2744000}; \; \rho_1'^3 = \frac{-82672615 - 244717902\sqrt{-3}}{2744000}.$$

N and N' being fractional and S moreover real and fractional, show that the three roots of this equation are real, rational and fractional; therefore the irreducible case occurs again in this example. From the fact that, when a, b, c are real, ρ, ρ' and ρ_1, ρ'_1 are conjugate complex numbers, and that, when moreover a, b, c are rational but fractional ρ, ρ' and ρ_1, ρ'_1 are of the form $\frac{\alpha + \beta\sqrt{-3}}{\gamma}$ and $\frac{\alpha - \beta\sqrt{-3}}{\gamma}$, it follows that $N = \rho\rho'$ and $N' = \rho_1\rho'_1$ are capable of being brought to the form $\frac{\alpha^2 + 3\beta^2}{\gamma^2}$, and that consequently the common denominators of ρ and ρ', as well as of ρ_1 and ρ'_1, are respectively equal to the square-roots of the denominators of N and N', when reduced to their lowest terms. Whence the denominator of the values of $\rho\rho'$ and $\rho_1\rho'_1$ must be equal to the product of the square-roots of the denominators of N and N'. Accordingly these denominators may be set aside in the determination of the quantities ρ, ρ', ρ_1, ρ'_1 by the process of "common factors," and supplied afterwards to the factors common to the numerators of those quantities.

Determination of the factor common:

$$\text{to } \rho\rho' = \frac{333481}{19600} \text{ and } \rho_1\rho'_1 = \frac{-286465 + 190146\sqrt{-3}}{19600}, \text{ viz. } \rho.$$

First operation: $-286465 + 190146\sqrt{-3} : 333481$

$$= -1 + \frac{47016 + 190146\sqrt{-3}}{333481} = -1 + 6\sqrt{-3}\,\frac{(31691 - 2612\sqrt{-3})}{333481};$$

second operation: $333481 : 31691 - 2612\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = \frac{31691}{3073}$ $\beta = \frac{2612}{3073}$; in integers nearly $\alpha = 10$; $\beta = 1$; remainder: $8735 - 557\sqrt{-3}$;

third operation: $31691 - 2612\sqrt{-3} : 8735 - 557\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = \frac{961}{508}$; $\beta = \frac{461}{508}$; nearly $\alpha = 2$; $\beta = 1$; remainder: $-2492 - 205\sqrt{-3}$;

fourth operation: $8735 - 557\sqrt{-3} : -2492 - 205\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = -\frac{55}{19}$; $\beta = \frac{47}{19}$; nearly $\alpha = -3$; $\beta = 2$; remainder: $29 - 1202\sqrt{-3}$;

fifth operation: $-2492 - 205\sqrt{-3} : 29 - 1202\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = \frac{2}{13}$; $\beta = -\frac{9}{13}$; nearly $\alpha = 0$; $\beta = -1$; remainder: $1114 - 176\sqrt{-3}$ $= 2(557 - 88\sqrt{-3})$;

sixth operation: $29 - 1202\sqrt{-3}$; $557 - 88\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = 1$; $\beta = -2$; *i. e.* $29 - 1202\sqrt{-3}$; $557 - 88\sqrt{-3} = 1 - 2\sqrt{-3}$;

therefore: $\overline{\frac{-286465 + 190146\sqrt{-3}}{19600} \Big| \frac{333481}{19600}} = \frac{557 - 88\sqrt{-3}}{140}$. Whence

$$\rho\rho' = \frac{(557 - 88\sqrt{-3})}{140}\frac{(557 + 88\sqrt{-3})}{140} = \frac{(-557 + 88\sqrt{-3})}{140}\frac{(-557 - 88\sqrt{-3})}{140};$$

$$\rho\rho'_1 = \frac{(557 - 88\sqrt{-3})}{140}\frac{(-629 + 242\sqrt{-3})}{140} = \frac{(-557 + 88\sqrt{-3})}{140}\frac{(629 - 242\sqrt{-3})}{140};$$

and after a verification similar to that in the preceding example, by comparing the cubes of $\frac{557 - 88\sqrt{-3}}{140}$ and $\frac{-557 + 88\sqrt{-3}}{140}$ with the value of ρ^3 $= \frac{3P - 2AN + 3S\sqrt{-3}}{2} = \frac{-133988021 + 79861320\sqrt{-3}}{2744000}$, we find

$$\rho = \frac{-557 + 88\sqrt{-3}}{140}; \quad \rho' = \frac{-557 - 88\sqrt{-3}}{140}; \quad \rho_1 = \frac{629 + 242\sqrt{-3}}{140};$$

$$\rho'_1 = \frac{629 - 242\sqrt{-3}}{140}.$$

Determination of the roots a, b, c:

$$a = \frac{-A + \rho + \rho'}{3} = \frac{-41 - 2\cdot557}{3\cdot140} = -\frac{11}{4} = -\frac{\rho_1 - \rho'_1}{\rho - \rho'} = -\frac{2\cdot242\sqrt{-3}}{2\cdot88\sqrt{-3}}$$

$$= -\frac{11}{4} = -\frac{2B - (\rho_1 + \rho'_1)}{2A + (\rho + \rho')} = -\frac{-1580 - 2\cdot 629}{82 - 2\cdot 557} = -\frac{11}{4} = -\frac{3C}{B + \rho_1 + \rho_1}$$

$$= -\frac{1287}{-790 + 2\cdot 629} = -\frac{11}{4};$$

$$b = \frac{-A + \theta'\rho + \theta\rho'}{3} = \frac{-2A - (\rho + \rho') - (\rho - \rho')\sqrt{-3}}{6} = \frac{-82 + 2\cdot 557 + 6\cdot 88}{6\cdot 140}$$

$$= \frac{13}{7} = -\frac{\theta'\rho_1 - \theta\rho'_1}{\theta'\rho - \theta\rho'} = -\frac{-(\rho_1 - \rho'_1) - (\rho_1 + \rho'_1)\sqrt{-3}}{-(\rho - \rho') - (\rho + \rho')\sqrt{-3}} = -\frac{-2\cdot 242\sqrt{-3} - 2\cdot 629\sqrt{-3}}{-2\cdot 88\sqrt{-3} + 2\cdot 557\sqrt{-3}}$$

$$= \frac{13}{7} = -\frac{2B - (\theta'\rho_1 + \theta\rho'_1)}{2A + (\theta'\rho + \theta\rho')} = -\frac{4B + (\rho_1 + \rho'_1) + (\rho_1 - \rho'_1)\sqrt{-3}}{4A - (\rho + \rho') - (\rho - \rho')\sqrt{-3}}$$

$$= -\frac{-3160 + 2\cdot 629 - 6\cdot 242}{164 + 2\cdot 557 + 6\cdot 88} = \frac{13}{7} = -\frac{3C}{B + \theta'\rho_1 + \theta\rho'_1}$$

$$= -\frac{6C}{2B - (\rho_1 + \rho'_1) - (\rho_1 - \rho'_1)\sqrt{-3}} = -\frac{6\cdot 429}{-1580 - 2\cdot 629 + 6\cdot 242} = \frac{13}{7};$$

$$c = \frac{-A + \theta\rho + \theta'\rho'}{3} = \frac{-2A - (\rho + \rho') + (\rho - \rho')\sqrt{-3}}{6} = \frac{-82 + 2\cdot 557 - 6\cdot 88}{6\cdot 140} = \frac{3}{5}$$

$$= -\frac{\theta\rho_1 - \theta'\rho'_1}{\theta\rho - \theta'\rho'} = -\frac{-(\rho_1 - \rho'_1) + (\rho_1 + \rho'_1)\sqrt{-3}}{-(\rho - \rho') + (\rho + \rho')\sqrt{-3}} = -\frac{-2\cdot 242\sqrt{-3} + 2\cdot 629\sqrt{-3}}{-2\cdot 88\sqrt{-3} - 2\cdot 557\sqrt{-3}}$$

$$= \frac{3}{5} = -\frac{2B - (\theta\rho_1 + \theta'\rho'_1)}{2A + (\theta\rho + \theta'\rho')} = -\frac{4B + (\rho_1 + \rho'_1) - (\rho_1 - \rho'_1)\sqrt{-3}}{4A - (\rho + \rho') + (\rho - \rho')\sqrt{-3}}$$

$$= -\frac{-3160 + 2\cdot 629 + 6\cdot 242}{164 + 2\cdot 557 - 6\cdot 88} = \frac{3}{5} = -\frac{3C}{B + \theta\rho_1 + \theta'\rho'_1} = -\frac{6C}{2B - (\rho_1 + \rho'_1) + (\rho_1 - \rho'_1)\sqrt{-3}}$$

$$= -\frac{6\cdot 429}{-1580 - 2\cdot 629 - 6\cdot 242} = \frac{3}{5}.$$

III.

$$x^3 + \frac{81}{55}x^2 + \frac{181}{385}x + \frac{3}{77} = 0;$$

$$A = \frac{81}{55};\ B = \frac{181}{385};\ C = \frac{3}{77};$$

$$N = \frac{16062}{21175} = \frac{112434}{148225};\ N' = \frac{7246}{148225};\ P = \frac{50652}{148225};\ S = \frac{932\sqrt{266}}{148225};$$

$$\rho\rho'_1 = \frac{25326 + 466\sqrt{266}\sqrt{-3}}{148225};\ \rho'\rho_1 = \frac{25326 - 466\sqrt{266}\sqrt{-3}}{148225};$$

$$\rho^3 = \frac{-34498548 + 538230\sqrt{266}\sqrt{-3}}{57066625};\ \rho'^3 = \frac{-34498548 - 538230\sqrt{266}\sqrt{-3}}{57066625};$$

$$\rho_1^3 = \frac{171856 + 20970\sqrt{266}\sqrt{-3}}{57066625};\ \rho'^3_1 = \frac{171856 - 20970\sqrt{266}\sqrt{-3}}{57066625}.$$

S being real but irrational shows that a, b, c are all real, and either all irrational or one rational and the other two irrational, as appears from the formula: $\left.\begin{matrix}a\\ b\end{matrix}\right\} = -\frac{2N' + (P \pm S)c}{2Nc + (P \mp S)}$, where N, N', P are rational and S irrational in the present case.

The composition of N, N', P in terms of a, b, c* shows, moreover, that in the latter case N and N' are capable of being brought to the form: $\frac{\alpha^2 + 3\lambda\beta^2}{\gamma^2} = \frac{\alpha + \beta\sqrt{\lambda}\sqrt{-3}}{\gamma} \cdot \frac{\alpha - \beta\sqrt{\lambda}\sqrt{-3}}{\gamma}$; ($\lambda$ being the radical of the irrational factor in S) and indeed:

$$N = \rho\rho' = \frac{126^2 + 3\cdot266\cdot11^2}{385^2} = \frac{(126 \pm 11\sqrt{266}\sqrt{-3})}{385}\,\frac{(126 \mp 11\sqrt{266}\sqrt{-3})}{385}$$

$$= \frac{(-126 \mp 11\sqrt{266}\sqrt{-3})}{385}\,\frac{(-126 \pm 11\sqrt{266}\sqrt{-3}}{385};\quad N' = \rho_1\rho'_1 = \frac{8^2 + 3\cdot266\cdot3^2}{385^2}$$

$$= \frac{(8 \pm 3\sqrt{266}\sqrt{-3})}{385}\,\frac{(8 \mp 3\sqrt{266}\sqrt{-3})}{385} = \frac{(-8 \mp 3\sqrt{266}\sqrt{-3})}{385}$$

$$\frac{(-8 \pm 3\sqrt{266}\sqrt{-3})}{385};\ \text{whence}\ \rho = \frac{126 \pm 11\sqrt{266}\sqrt{-3}}{385}\ \text{or} = \frac{-126 \mp 11\sqrt{266}\sqrt{-3}}{385};$$

$$\rho' = \frac{126 \mp 11\sqrt{266}\sqrt{-3}}{385}\ \text{or} = \frac{-126 \pm 11\sqrt{266}\sqrt{-3}}{385};\ \rho_1 = \frac{8 \pm 3\sqrt{266}\sqrt{-3}}{385}$$

$$\text{or} = \frac{-8 \mp 3\sqrt{266}\sqrt{-3}}{385};\ \rho'_1 = \frac{8 \mp 3\sqrt{266}\sqrt{-3}}{385}\ \text{or} = \frac{-8 \pm 3\sqrt{266}\sqrt{-3}}{385}.$$

Now as the cubes of $\frac{126 - 11\sqrt{266}\sqrt{-3}}{385}$, $\frac{126 + 11\sqrt{266}\sqrt{-3}}{385}$, $\frac{-8 - 3\sqrt{266}\sqrt{-3}}{385}$, $\frac{-8 + 3\sqrt{266}\sqrt{-3}}{385}$ are identical with the above values of ρ^3, ρ'^3, ρ_1^3, $\rho_1'^3$, it is evident that these numbers are the values of ρ, ρ', ρ_1, ρ'_1, respectively.

Determination of the roots a, b, c:

$$a = \frac{-A + \rho + \rho'}{3} = \frac{-7\cdot81 + 2\cdot126}{3\cdot385} = -\frac{3}{11} = -\frac{\rho_1 - \rho'_1}{\rho - \rho'} = -\frac{-2\cdot3\sqrt{266}\sqrt{-3}}{-2\cdot11\sqrt{266}\sqrt{-3}}$$

$$= -\frac{3}{11} = -\frac{2B - (\rho_1 + \rho'_1)}{2A + (\rho + \rho')} = -\frac{2\cdot181 + 2\cdot8}{2\cdot7\cdot81 + 2\cdot126} = -\frac{3}{11} = -\frac{3C}{B + \rho_1 + \rho'_1}$$

$$= -\frac{5\cdot3\cdot3}{181 - 2\cdot8} = -\frac{3}{11};$$

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* See note, page 33.

$$b = \frac{-A + \theta'\rho + \theta\rho'}{3} = \frac{-2A - (\rho + \rho') - (\rho - \rho')\sqrt{-3}}{6}$$

$$= \frac{-2\cdot7\cdot81 - 2\cdot126 + 2\cdot11\sqrt{266}\sqrt{-3}\sqrt{-3}}{6\cdot385} = -\frac{21 + \sqrt{266}}{35} = -\frac{3\sqrt{7} + \sqrt{38}}{5\sqrt{7}}$$

$$= -\frac{\theta'\rho_1 - \theta\rho'_1}{\theta'\rho - \theta\rho'} = -\frac{-(\rho_1 - \rho'_1) - (\rho_1 + \rho'_1)\sqrt{-3}}{-(\rho - \rho') - (\rho + \rho')\sqrt{-3}} = -\frac{2\cdot3\sqrt{266}\sqrt{-3} + 2\cdot8\sqrt{-3}}{2\cdot11\sqrt{266}\sqrt{-3} - 2\cdot12\sqrt{-3}}$$

$$= -\frac{21 + \sqrt{266}}{35} = -\frac{3\sqrt{7} + \sqrt{38}}{5\sqrt{7}} = -\frac{2B - (\theta'\rho_1 + \theta\rho'_1)}{2A + (\theta'\rho + \theta\rho')}$$

$$= -\frac{4B + (\rho_1 + \rho'_1) + (\rho_1 - \rho'_1)\sqrt{-3}}{4A - (\rho + \rho') - (\rho - \rho')\sqrt{-3}} = -\frac{4\cdot181 - 2\cdot8 - 2\cdot3\sqrt{266}\sqrt{-3}\sqrt{-3}}{4\cdot7\cdot81 - 2\cdot126 + 2\cdot11\sqrt{266}\sqrt{-3}\sqrt{-3}}$$

$$= -\frac{118 + 3\sqrt{266}}{336 - 11\sqrt{266}} = -\frac{21 + \sqrt{266}}{35} = -\frac{3\sqrt{7} + \sqrt{38}}{5\sqrt{7}} = -\frac{3C}{B + \theta'\rho_1 + \theta\rho'_1}$$

$$= -\frac{6C}{2B - (\rho_1 + \rho'_1) - (\rho_1 - \rho'_1)\sqrt{-3}} = -\frac{5\cdot18}{2\ 181 + 2\cdot8 + 2\cdot3\sqrt{266}\sqrt{-3}\sqrt{-3}}$$

$$= -\frac{5}{21 - \sqrt{266}} = -\frac{21 + \sqrt{266}}{35} = -\frac{3\sqrt{7} + \sqrt{38}}{5\sqrt{7}};$$

$$c = \frac{-A + \theta\rho + \theta'\rho'}{3} = \frac{-2A - (\rho + \rho') + (\rho - \rho)\sqrt{-3}}{6}$$

$$= \frac{-2\cdot7\cdot81 - 2\ 126 - 2\cdot11\sqrt{266}\sqrt{-3}}{6\cdot385} = -\frac{21 - \sqrt{266}}{35} = -\frac{3\sqrt{7} - \sqrt{38}}{5\sqrt{7}}$$

$$= -\frac{\theta\rho_1 - \theta'\rho'_1}{\theta\rho - \theta'\rho'} = -\frac{-(\rho_1 - \rho'_1) + (\rho_1 + \rho'_1)\sqrt{-3}}{-(\rho - \rho') + (\rho + \rho')\sqrt{-3}} = -\frac{2\cdot3\sqrt{266}\sqrt{-3} - 2\cdot8\sqrt{-3}}{2\cdot11\sqrt{266}\sqrt{-3} + 2\cdot126\sqrt{-3}}$$

$$= -\frac{21 - \sqrt{266}}{35} = -\frac{3\sqrt{7} - \sqrt{38}}{5\sqrt{7}} = -\frac{2B - (\theta\rho_1 + \theta'\rho'_1)}{2A + (\theta\rho + \theta'\rho')}$$

$$= -\frac{4B + (\rho_1 + \rho'_1) - (\rho_1 - \rho'_1)\sqrt{-3}}{4A - (\rho + \rho') + (\rho - \rho')\sqrt{-3}} = -\frac{4\cdot181 - 2\cdot8 + 2\cdot3\sqrt{266}\sqrt{-3}\sqrt{-3}}{4\cdot7\cdot81 - 2\cdot126 - 2\cdot11\sqrt{266}\sqrt{-3}\sqrt{-3}}$$

$$= -\frac{118 - 3\sqrt{266}}{336 + 11\sqrt{266}} = -\frac{21 - \sqrt{266}}{35} = -\frac{3\sqrt{7} - \sqrt{38}}{5\sqrt{7}} = -\frac{3C}{B + \theta\rho_1 + \theta'\rho'_1}$$

$$= -\frac{6C}{2B - (\rho_1 + \rho'_1) + (\rho_1 - \rho'_1)\sqrt{-3}} = -\frac{5\cdot18}{2\cdot181 + 2\cdot8 - 2\cdot3\sqrt{266}\sqrt{-3}\sqrt{-3}}$$

$$= -\frac{5}{21 + \sqrt{266}} = -\frac{21 - \sqrt{266}}{35} = -\frac{3\sqrt{7} - \sqrt{38}}{5\sqrt{7}}$$

APPENDIX.

The following table of the values of $\alpha^2 + 3\beta^2$, corresponding to all integral values of α and β from 0 to 9, will be found convenient for the determination of $\rho, \rho', \rho_1, \rho'_1$ by the decomposition of $N = \rho\rho'$ and $N' = \rho_1\rho'_1$ into their factors ρ, ρ' and ρ_1, ρ'_1 in case N and N' are tractable numbers.

	$\alpha = 0$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$	$\alpha = 6$	$\alpha = 7$	$\alpha = 8$	$\alpha = 9$
$\beta = 0$	$0^2+3.0^2= 0$	$1^2+3.0^2= 1$	$2^2+3.0^2= 4$	$3^2+3.0^2= 9$	$4^2+3.0^2= 16$	$5^2+3.0^2= 25$	$6^2+3.0^2= 36$	$7^2+3.0^2= 49$	$8^2+3.0^2= 64$	$9^2+3.0^2= 81$
$\beta = 1$	$0^2+3\ 1^2= 3$	$1^2+3.1^2= 4$	$2^2+3.1^2= 7$	$3^2+3.1^2= 12$	$4^2+3.1^2= 19$	$5^2+3.1^2= 28$	$6^2+3.1^2= 39$	$7^2+3.1^2= 52$	$8^2+3.1^2= 67$	$9^2+3.1^2= 84$
$\beta = 2$	$0^2+3.2^2= 12$	$1^2+3.2^2= 13$	$2^2+3\ 2^2= 16$	$3^2+3.2^2= 21$	$4^2+3.2^2= 28$	$5^2+3.2^2= 37$	$6^2+3.2^2= 48$	$7^2+3.2^2= 61$	$8^2+3.2^2= 76$	$9^2+3.2^2= 93$
$\beta = 3$	$0^2+3.3^2= 27$	$1^2+3.3^2= 28$	$2^2+3.3^2= 31$	$3^2+3.3^2= 36$	$4^2+3.3^2= 43$	$5^2+3.3^2= 52$	$6^2+3.3^2= 63$	$7^2+3.3^2= 76$	$8^2+3.3^2= 91$	$9^2+3.3^2=108$
$\beta = 4$	$0^2+3.4^2= 48$	$1^2+3.4^2= 49$	$2^2+3.4^2= 52$	$3^2+3\ 4^2= 57$	$4^2+3.4^2= 64$	$5^2+3.4^2= 73$	$6^2+3.4^2= 84$	$7^2+3.4^2= 97$	$8^2+3.4^2=112$	$9^2+3.4^2=129$
$\beta = 5$	$0^2+3\ 5^2= 75$	$1^2+3.5^2= 76$	$2^2+3\ 5^2= 79$	$3^2+3.5^2= 84$	$4^2+3.5^2= 91$	$5^2+3.5^2=100$	$6^2+3.5^2=111$	$7^2+3.5^2=124$	$8^2+3.5^2=139$	$9^2+3.5^2=156$
$\beta = 6$	$0^2+3.6^2=108$	$1^2+3\ 6^2=109$	$2^2+3.6^2=112$	$3^2+3.6^2=117$	$4^2+3.6^2=124$	$5^2+3.6^2=133$	$6^2+3.6^2=144$	$7^2+3.6^2=157$	$8^2+3.6^2=172$	$9^2+3.6^2=189$
$\beta = 7$	$0^2+3.7^2=147$	$1^2+3.7^2=148$	$2^2+3.7^2=151$	$3^2+3.7^2=156$	$4^2+3.7^2=163$	$5^2+3.7^2=172$	$6^2+3.7^2=183$	$7^2+3.7^2=196$	$8^2+3.7^2=211$	$9^2+3.7^2=228$
$\beta = 8$	$0^2+3.8^2=192$	$1^2+3.8^2=193$	$2^2+3.8^2=196$	$3^2+3.8^2=201$	$4^2+3.8^2=208$	$5^2+3.8^2=217$	$6^2+3.8^2=228$	$7^2+3.8^2=241$	$8^2+3.8^2=256$	$9^2+3.8^2=273$
$\beta = 9$	$0^2+3.9^2=243$	$1^2+3.9^2=244$	$2^2+3.9^2=247$	$3^2+3.9^2=252$	$4^2+3.9^2=259$	$5^2+3.9^2=268$	$6^2+3.9^2=279$	$7^2+3.9^2=292$	$8^2+3.9^2=307$	$9^2+3.9^2=324$